

Dynamical systems method (DSM) for general nonlinear equations.

A.G. Ramm

Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

If $F : H \rightarrow H$ is a map in a Hilbert space H , $F \in C^2_{loc}$, and there exists y , such that $F(y) = 0$, $F'(y) \neq 0$, then equation $F(u) = 0$ can be solved by a DSM (dynamical systems method). This method yields also a convergent iterative method for finding y , and this method converges at the rate of a geometric series. It is not assumed that y is the only solution to $F(u) = 0$. Stable approximation to a solution of the equation $F(u) = f$ is constructed by a DSM when f is unknown but f_δ is known, where $\|f_\delta - f\| \leq \delta$.

1 Introduction

In this paper a method for solving fairly general class of nonlinear operator equations $F(u) = 0$ in a Hilbert space is proposed, its convergence is proved, and an iterative method for solving the above equation is constructed. Convergence of the iterative method is also proved. These results are based on the following assumptions: a) the above equation has a solution y , possibly non-unique, b) $F \in C^2_{loc}$, and c) $F'(y) \neq 0$. No restrictions on the rate of growth of nonlinearity are made. The literature on methods for solving nonlinear equations is large (see, e.g., [1] and references therein). Most of the results obtained so far are based on Newton-type methods and their modifications. There is also a well-developed theory for equations with monotone operators ([3]). The method used in this paper is a version of the dynamical systems method which is studied in [2]. The idea of this method is described briefly below.

Let $F : H \rightarrow H$ be a map in a Hilbert space. Assume that equation

$$F(u) = 0 \tag{1}$$

MSC: 47H15, 47H20, 65H10, 65J15, 65N20

key words: nonlinear operator equations, iterative methods, DSM-dynamical systems method

has a solution y , possibly non-unique, and

$$F'(y) \neq 0, \quad (2)$$

where F' is the Fréchet derivative of F . This assumption means that $F'(y)$ is not equal to zero identically on H . Thus, it is a weak assumption. Assume that $F \in C^2_{loc}$, i.e.,

$$\sup_{u \in B(u_0, R)} \|F^{(j)}(u)\| \leq M_j(R) \quad 0 \leq j \leq 2, \quad (3)$$

where $u_0 \in H$ is a given element, $R > 0$, and no restrictions on the growth of $M_j(R)$ as R grows are made. This means that the nonlinearity F can grow arbitrarily fast as $\|u - u_0\|$ grows. It is known that under these assumptions equation (1) may have no solutions. Thus, we have assumed that a solution y to (1) exists.

We do not assume that $F'(u)$ has a bounded inverse operator. Therefore the standard Newton-type methods are not applicable. Dynamical system method (DSM) consists of finding an operator Φ such that the problem

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0 \quad (4)$$

has a unique global solution $u(t)$, there exists $u(\infty)$, and $F(u(\infty)) = 0$. To ensure the unique local solvability of (4) we assume that

$$\|\Phi(t, u) - \Phi(t, v)\| \leq L(R)\|u - v\| \quad \forall u, v \in B(u_0, R).$$

Then the global existence of the unique local solution holds if $\sup_t \|u(t)\| < \infty$.

The results of this paper can be summarized in two theorems. Let us denote

$$A := F'(u(t)), \quad T := A^*A, \quad T_a := T + aI; \quad \tilde{A} := F'(y), \quad \tilde{T} = \tilde{A}^*\tilde{A}. \quad (5)$$

Assume that $a(t)$ is a positive monotonically decaying function,

$$a(t) > 0, \quad \lim_{t \rightarrow \infty} a(t) = 0, \quad \frac{|\dot{a}|}{a} \leq \frac{1}{2}. \quad (6)$$

Theorem 1. *If a solution y to equation (1) exists, possibly is non-unique, and assumptions (2) and (3) hold, then $y = u(\infty)$, where $u(t)$ solves the following DSM problem:*

$$\dot{u} = -T_{a(t)}^{-1}[A^*F(u(t)) + a(t)(u(t) - z)], \quad u(0) = u_0, \quad (7)$$

and where z and u_0 are suitably chosen.

Theorem 2. *Under the assumptions of Theorem 1, the iterative process*

$$u_{n+1} = u_n - h_n T_{a_n}^{-1}[A^*(u_n)F(u_n) + a_n(u_n - z)], \quad u_0 = u_0, \quad (8)$$

where $h_n > 0$ and $a_n > 0$ are suitably chosen, generates the sequence u_n converging to y .

Remark 1. The suitable choices of a_n and h_n are discussed in the proof of Theorem 2.

Remark 2. Essentially, Theorem 1 says that any solvable operator equation with C_{loc}^2 operator, satisfying only a weak assumption (2), can be solved by a DSM. Condition (2) means that the range of the linear operator $F'(y)$ contains at least one non-zero element. It allows $F'(y)$ to have infinite-dimensional null-space.

In Section 2 we prove Theorem 1 and Theorem 2. In their proofs we use the following lemmas.

Lemma 1. Assume that $g(t) \geq 0$ is a $C^1([0, \infty])$ function satisfying the inequality

$$\dot{g} \leq -\gamma(t)g + \alpha(t)g^2 + \beta(t), \quad t \geq 0, \quad \dot{g} := \frac{dg}{dt}, \quad (9)$$

where γ , α and β are nonnegative continuous functions. Assume that there exists $\mu \in C^1([0, \infty))$, $\mu > 0$, $\lim_{t \rightarrow \infty} \mu(t) = \infty$, such that

$$i) \alpha(t) \leq \frac{\mu(t)}{2} \left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)} \right), \quad ii) \beta \leq \frac{1}{2\mu(t)} \left(\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)} \right), \quad iii) g(0)\mu(0) < 1. \quad (10)$$

Then any solution to (9) exists on $[0, \infty)$ and

$$0 \leq g(t) < \frac{1}{\mu(t)}, \quad t \in [0, \infty). \quad (11)$$

Lemma 1 is proved in [2, pp.66-70].

Lemma 2. Let $g_{n+1} \leq \gamma g_n + pg_n^2$, $g_0 := m > 0$, $0 < \gamma < 1$, $p > 0$. If $m < \frac{q-\gamma}{p}$, where $\gamma < q < 1$, then $\lim_{n \rightarrow \infty} g_n = 0$, and $g_n \leq g_0 q^n$.

Proof of Lemma 2. Estimate $g_1 \leq \gamma m + pm^2 \leq qm$ holds if $m \leq \frac{q-\gamma}{p}$, $\gamma < q < 1$. Assume that $g_n \leq g_0 q^n$. Then

$$g_{n+1} \leq \gamma g_n q^n + p(g_0 q^n)^2 = g_0 q^n (\gamma + pg_0 q^n) < g_0 q^{n+1},$$

because $\gamma + pg_0 q^n < \gamma + pg_0 q \leq q$. Lemma 2 is proved. \square

2 Proofs

Proof of Theorem 1. If $F'(y) := A$ is linear and $A \neq 0$, then there exists $v_1 \neq 0$, $v_1 = \tilde{T}v$. By the linearity of \tilde{T} , every element cv_1 belongs to the range of \tilde{T} for any constant c , because $cv_1 = \tilde{T}(cv)$. Therefore there exists a z such that $y - z = \tilde{T}v$, where $\|v\| > 0$ can be chosen arbitrarily small. How small $\|v\|$ should be chosen will become clear later. Let $u(t) - y := w(t)$, $\|w(t)\| := g(t)$. Write equation (7) as

$$\dot{w} = -T_{a(t)}^{-1} [A^*(F(u) - F(y)) + a(t)w + a(t)(y - z)], \quad (12)$$

and use the formula $F(u) - F(y) = Aw + K$, where $\|K\| \leq \frac{M_2 g^2}{2}$. Then

$$\dot{w} = -w - T_{a(t)}^{-1} A^* K - a(t) T_{a(t)}^{-1} \tilde{T} v. \quad (13)$$

Multiply this equation by w in H and use the estimate $\|T_a^{-1} A^*\| \leq \frac{1}{2\sqrt{a}}$, $a > 0$, to get

$$g\dot{g} \leq -g^2 + \frac{1}{2\sqrt{a(t)}} \frac{M_2 g^2}{2} + a(t) \left\| \left(T_{a(t)}^{-1} - \tilde{T}_{a(t)}^{-1} + \tilde{T}_{a(t)}^{-1} \right) \tilde{T} \right\| \|v\|.$$

If $a > 0$ then $\|\tilde{T}_a^{-1} \tilde{T}\| \leq 1$,

$$a\|T_a^{-1}\| \leq 1, \quad a\|(T_a^{-1} - \tilde{T}_a^{-1})\tilde{T}\| = a\|T_a^{-1}(A^* A - \tilde{A}^* \tilde{A})\tilde{T}_a^{-1}\tilde{T}\| \leq 2M_1 M_2 g.$$

Collecting the above estimates and choosing $\|v\|$ so that $2M_1 M_2 \|v\| \leq \frac{1}{2}$, we obtain

$$\dot{g} \leq -\frac{g}{2} + \frac{c_0 g^2}{\sqrt{a(t)}} + a(t)\|v\|, \quad c_0 := \frac{M_2}{4}. \quad (14)$$

Apply Lemma 1 to (14). Here $\gamma = \frac{1}{2}$, $\alpha = \frac{c_0}{\sqrt{a(t)}}$, $\beta = a(t)\|v\|$. Let $\mu(t) = \frac{\lambda}{\sqrt{a(t)}}$, $\lambda = \text{const} > 0$. Condition *i*) of Lemma 1 holds if $\frac{c_0}{\sqrt{a(t)}} \leq \frac{\lambda}{2\sqrt{a(t)}} \left(\frac{1}{2} - \frac{1}{2} \frac{|\dot{a}|}{a} \right)$. This inequality holds if $\lambda \geq 8c_0$, see the last assumption (6). Condition *iii*) holds if $g(0) \frac{\lambda}{\sqrt{a(0)}} < 1$. This inequality holds (for any initial value $g(0) = \|u_0 - y\|$) if $a(0)$ is sufficiently large. Condition *ii*) holds if $\sqrt{a(t)}\|v\| \leq \frac{1}{8\lambda}$, where we have used the last assumption (6) again. This inequality holds if $(*) \quad 8\lambda\sqrt{a(0)}\|v\| \leq 1$. (Recall that $a(0) \geq a(t)$ due to monotonicity of $a(t)$.) Inequality $(*)$ holds if $\|v\|$ is sufficiently small. Thus, if $\|v\|$ is sufficiently small, then, by Lemma 1, we get $g(t) < \frac{\sqrt{a(t)}}{\lambda}$, so $\|u(t) - y\| \leq \frac{\sqrt{a(t)}}{\lambda} \rightarrow 0$ as $t \rightarrow \infty$.

Theorem 1 is proved. \square

Proof of Theorem 2. Let $w_n := u_n - y$, $g_n := \|w_n\|$. As in the proof of Theorem 1, we assume $2M_1 M_2 \|v\| \leq \frac{1}{2}$ and rewrite (8) as

$$w_{n+1} = w_n - h_n T_{a_n}^{-1} [A^*(u_n)(F(u_n) - F(y)) + a_n w_n + a_n(y - z)], \quad w_0 = \|u_0 - y\|.$$

Using the Taylor formula

$$F(u_n) - F(y) = A(u_n)w_n + K(w_n), \quad \|K\| \leq \frac{M_2 g_n^2}{2},$$

the estimate $\|T_{a_n}^{-1} A^*(u_n)\| \leq \frac{1}{2\sqrt{a_n}}$, and the formula $y - z = \tilde{T}v$, we get

$$w_{n+1} = (1 - h_n)w_n - h_n T_{a_n}^{-1} A^*(u_n)K(w_n) - h_n a_n T_{a_n}^{-1} \tilde{T} v. \quad (15)$$

Taking into account that $\|\tilde{T}_a^{-1}\tilde{T}\| \leq 1$, and $a\|T_a^{-1}\| \leq 1$ if $a > 0$, we obtain

$$\|T_{a_n}^{-1}\tilde{T}v\| \leq \|(T_{a_n}^{-1} - \tilde{T}_{a_n}^{-1})\tilde{T}\| \|v\| + \|v\|,$$

and

$$\|(T_{a_n}^{-1} - \tilde{T}_{a_n}^{-1})\tilde{T}\| = \|T_{a_n}^{-1}(\tilde{T}_{a_n} - T_{a_n})\tilde{T}_{a_n}^{-1}\tilde{T}\| \leq \frac{2M_1M_2g_n}{a_n} := \frac{c_1g_n}{a_n}.$$

Let $c_0 := \frac{M_2}{4}$. Then we obtain from (15) the following inequality:

$$g_{n+1} \leq (1 - h_n)g_n + \frac{c_0h_ng_n^2}{\sqrt{a_n}} + c_1h_n\|v\|g_n + h_na_n\|v\|.$$

We have assumed in the proof of Theorem 1 that $c_1\|v\| \leq \frac{1}{2}$. Thus

$$g_{n+1} \leq (1 - \frac{h_n}{2})g_n + \frac{c_0h_n}{\sqrt{a_n}}g_n^2 + h_na_n\|v\|.$$

Choose $a_n = 16c_0^2g_n^2$. Then $\frac{c_0g_n}{\sqrt{a_n}} = \frac{1}{4}$, and

$$g_{n+1} \leq (1 - \frac{h_n}{4})g_n + 16c_0h_n\|v\|g_n^2, \quad g_0 = \|u_0 - y\| \leq R, \quad (16)$$

where $R > 0$ is defined in (3). Take $h_n = h \in (0, 1)$ and choose $g_0 := m$ such that $m < \frac{q+h-1}{16c_0h\|v\|}$, where $q \in (0, 1)$ and $q + h > 1$. Then Lemma 2 implies

$$\|u_n - y\| \leq g_0q^n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Theorem 2 is proved. □

3 Stability of the solution

Assume that $F(y) = f$, where the exact data f are not known but the noisy data f_δ are given, $\|f_\delta - f\| \leq \delta$. Then the DSM yields a stable approximation of the solution y if the stopping time t_δ is properly chosen. The DSM is similar to (7):

$$\dot{u}_\delta = -T_{a(t)}^{-1}[A^*(F(u_\delta(t)) - f_\delta) + a(t)(u_\delta(t) - z)], \quad u_\delta(0) = u_0, \quad (17)$$

Let $w_\delta := u_\delta(t) - y$, $g_\delta(t) := \|w_\delta\|$. As in the proof of Theorem 1 we derive the inequality similar to (14):

$$\dot{g}_\delta \leq -\frac{g_\delta}{2} + \frac{c_0g_\delta^2}{\sqrt{a(t)}} + a(t)\|v\| + \frac{\delta}{2\sqrt{a(t)}}, \quad c_0 := \frac{M_2}{4}, \quad (18)$$

and apply Lemma 1. The only difference is in checking condition *ii*) of Lemma 1. This condition now takes the form:

$$a(t)\|v\| + \frac{\delta}{2\sqrt{a(t)}} \leq \frac{\sqrt{a(t)}}{8\lambda}.$$

This condition can only be satisfied for $t \in [0, t_\delta]$, where $t_\delta < \infty$. The stopping time t_δ can be determined, for example, from the equation $4\lambda \frac{\delta}{a(t)} = \frac{1}{2}$, provided that v is chosen sufficiently small, so that $8\lambda\sqrt{a(0)}\|v\| \leq \frac{1}{2}$, and $\lambda \geq 8c_0$ as in the proof of Theorem 1. Then, by Lemma 1, we have $g_\delta(t_\delta) < \frac{\sqrt{a(t)}}{\lambda} \rightarrow 0$ as $\delta \rightarrow 0$. Let us formulate the result.

Theorem 3. *Let $u_\delta := u_\delta(t_\delta)$, where $u_\delta(t)$ solves problem (17) and t_δ is chosen as above. Then $\lim_{\delta \rightarrow 0} \|u_\delta - y\| = 0$.*

References

- [1] Ortega, J., Rheinboldt, W., Iterative solution of nonlinear equations in several variables, SIAM, Philadelphia, 2000.
- [2] Ramm, A. G. , Inverse problems, Springer, New York, 2005.
- [3] Zeidler, E., Nonlinear functional analysis and its applications, II-B, Springer, New York, 1985.